

L. Szpiro's conjecture on Gorenstein algebras in codimension 2

Christian Böhning

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Abstract

A Gorenstein A -algebra R of codimension 2 is a perfect finite A -algebra such that $R \cong \operatorname{Ext}_A^2(R, A)$ holds as R -modules, A being a Cohen-Macaulay local ring with $\dim A - \dim_A R = 2$.

I prove a structure theorem for these algebras improving on an old theorem of M. Grassi [Gra]. Special attention is paid to the question how the ring structure of R is encoded in its Hilbert resolution. It is shown that R is automatically a ring once one imposes a very weak depth condition on a determinantal ideal derived from a presentation matrix of R over A . Furthermore, the interplay of Gorenstein algebras and Koszul modules as introduced by M. Grassi is clarified. I include graded analogues of the afore-mentioned results when possible.

Questions of applicability to the theory of surfaces of general type (namely, canonical surfaces in \mathbb{P}^4) have served as a guideline in these commutative algebra investigations.

0 Introduction and statement of results

Motivation for investigating the types of questions treated in the present article sprang from two different, though closely related sources, the first algebro-geometric, the second one purely algebraic in spirit. Though in this paper I will restrict myself to touching upon the latter only, both themes can, I think, be best understood in conjunction, so I will briefly discuss them jointly in the introduction.

From the point of view of algebraic geometry, the perhaps earliest traces of the story may be located in the book [En] by F. Enriques where he treats the structure of canonical surfaces in \mathbb{P}^4 with $q = 0$, $p_g = 5$, $K^2 = 8$ and 9 (cf. loc. cit., p. 284ff. ; they are the complete intersections of type $(2, 4)$ and $(3, 3)$; the word “canonical” means that for those surfaces the 1-canonical

map gives a birational morphism onto the image in \mathbb{P}^4). Subsequently the case $K^2 = 10$ was solved by C. Ciliberto (cf. [Cil]) using liaison arguments, and along different lines, D. Roßberg (cf. [Roß]) tackled the problem for $K^2 = 11$ and $= 12$, the question being open for higher values of K^2 .

Translated into the language of commutative and homological algebra, the first difficulty to face in an attempt to treat cases with higher K^2 is to find a satisfactory structure theorem for Gorenstein algebras in codimension 2; roughly, these are finite A -algebras R (A some “nice” base ring) with $R \cong \operatorname{Ext}_A^2(R, A)$, possibly up to twist if the base ring is graded (cf. section 2 below for precise definitions). The connection with the above surfaces is established by remarking that their canonical rings are codimension 2 Gorenstein algebras over the homogeneous coordinate ring of \mathbb{P}^4 .

With regard to a structure theorem, the above qualifier “satisfactory” means precisely that one should be able to tell from practically verifiable and non-tautological conditions how the Hilbert resolution of R over A encodes (1) the “duality” $R \cong \operatorname{Ext}_A^2(R, A)$ and (2) the fact that R has not only an A -module structure, but also a ring structure. Whereas (1) is by now fairly well understood, (2) is not, and the main purpose of this paper is to show how (2) can be disposed with. Let me mention at this point that, in the course of his investigations concerning low rank vector bundles on projective spaces, Lucien Szpiro was presumably the first to point out the need for and formulate some conjectural statements concerning a good structure theorem for Gorenstein algebras in codimension 2, which is why his name appears in the title of this paper.

Since then, quite a good deal of work has been done on this problem, in geometric and algebraic guises. Let me therefore give some perspective on its history: In [Cat2] canonical surfaces in \mathbb{P}^3 are studied (from a moduli point of view) via a structure theorem proved therein for Gorenstein algebras in codimension 1. It is shown that the duality $R \cong \operatorname{Ext}_A^1(R, A)$ for these algebras translates into the fact that the Hilbert resolution of R can be chosen to be self-dual; moreover, that the presence of a ring structure on R is equivalent to a (closed) condition on the Fitting ideals of a presentation matrix of R as A -module (the so-called “ring condition” or “rank condition” or “condition of Rouché-Capelli”, abbreviated R.C. in any case). These ideas were developed further and generalized in [M-P] and [dJ-vS] (within the codimension 1 setting). In particular the latter papers show that R.C. can be rephrased in terms of annihilators of elements of R and gives a good structure theorem also in the non-Gorenstein case. On the base of all this, M. Grassi turned attention to the codimension 2 setting. In [Gra] he isolated the abstract kernel of the problem and proved that also for codimension 2

Gorenstein algebras the duality $R \cong \text{Ext}_A^2(R, A)$ is equivalent to R having a self-dual resolution. He introduced the concept of Koszul modules which provide a nice framework for dealing with Gorenstein algebras and also proposed a structure theorem for the codimension 2 case. Unfortunately, as for the question how the ring structure of a codimension 2 Gorenstein algebra is encoded in its Hilbert resolution, the conditions he gives are tautological and (therefore) too complicated (although they are necessary and sufficient). More recently, D. Eisenbud and B. Ulrich (cf. [E-U]) re-examined the ring condition and gave a generalization of it which appears to be more natural than the direction in which [Gra] is pointing. But essentially, they only give sufficient conditions for R to be a ring, and these are not fulfilled in the applications to canonical surfaces one has in mind. More information on the development sketched here can be found in [Cat4]. For a deeper study of that part of the story that originates from the duality $R \cong \text{Ext}_A^2(R, A)$ and its effects on the symmetry properties of the Hilbert resolution of R , as well as for a generalization of this to the bundle case cf. [E-P-W].

In the present article, on the contrary, the approach to the problem of detecting the ring structure of R in its Hilbert resolution is based on a philosophy already present in [Cat4], cf. p. 48: The ring condition R.C. is automatical under some mild extra condition which, moreover, works well in the applications to canonical surfaces (the condition corresponds to the requirement that the canonical image in \mathbb{P}^4 should have only isolated singularities). This is the content of theorem 1.1 (and its graded analogue theorem 1.3) below. The idea of proof is marvellously simple: One considers the sheaf \mathcal{R} associated to R on $X := \text{Spec } A$ and uses the fact that “the locus where \mathcal{R} is not known to be a sheaf of rings a priori is small”, i.e. if Y is the support of \mathcal{R} , there is an open $U \subset Y$ such that the complement $Z := Y \setminus U$ has codimension 2 in Y and $\mathcal{R}|_U$ is known to be a sheaf of rings from elementary considerations. Then using the fact that R is Cohen-Macaulay and some local cohomology computations, one can easily check that the ring structure extends from $\mathcal{R}|_U$ to \mathcal{R} . The situation is thus very much reminiscent of that encountered in the familiar Hartogs theorem in several complex variables.

Theorem 1.1 works without the Gorenstein condition on R which will enter only in section 2. There I basically only combine the results from section 1 with the symmetry statements known for the resolutions of Gorenstein algebras from [Gra]. I have chosen to dwell on the proofs of the latter, partly because occasionally minor simplifications could be made, partly for the sake of completeness. Theorem 2.4 (resp. theorem 2.5) contains the characterization of Gorenstein algebras in codimension 2.

Section 3 is included to provide a strengthening of theorem 2.4 (the local

case of the structure theorem). It is also meant to clarify the relationship between Gorenstein algebras in codimension 2 and Koszul modules as introduced by M. Grassi in [Gra]. It is shown that a Gorenstein algebra over a local ring (always assumed to be Cohen-Macaulay) admits a Gorenstein-symmetric resolution which is at the same time of Koszul module type (see section 3 for precise statements).

Finally, there is reasonable hope that the techniques developed in this paper may facilitate the study of canonical surfaces in \mathbb{P}^4 with $p_g = 5$, $q = 0$, $K^2 \geq 13$. In fact I propose to apply them as well to questions of existence as to the problem of describing their moduli. Tangible results have already been obtained and will be published elsewhere.

It is a pleasure for me to thank Fabrizio Catanese for posing the problem and many useful discussions.

1 How the ring structure is encoded

The following result should be viewed as a rather general extension theorem giving conditions under which a module that is a ring "in codimension 1" is already itself a ring, and giving a description of the resulting structure.

Theorem 1.1. *Let A be a Cohen-Macaulay local ring with maximal ideal \mathfrak{m} and residue field k and let R be a finite A -module.*

Let a length 2 minimal free resolution

$$0 \longrightarrow F_2 \xrightarrow{\psi} F_1 \xrightarrow{\varphi} F_0 \xrightarrow{p} R \longrightarrow 0$$

of R be given. View φ as a presentation matrix of R and let φ' be the matrix φ with first row erased. Denote by I resp. I' the zeroth Fitting ideals (=ideals of maximal minors) of φ resp. φ' .

Suppose that $\text{depth}(\text{Ann}_A R, A) (= \text{codim}_A R) = 2$, the maximum possible in view of the inequality $\text{depth}(\text{Ann}_A R, A) \leq \text{prodim}_A R$.

Then, if moreover the following condition holds

$$(\heartsuit) \text{ depth } I' \geq 4,$$

it follows that $R \cong \text{Hom}_{A_Y}(C, C)$, where $A_Y := A/\text{Ann}_A R$ and $C := \text{Hom}_{A_Y}(R, A_Y)$ is the so-called conductor of R into A_Y ; and R is given a (commutative) ring structure if we define multiplication as composition of endomorphisms of C . The identity element of R is the image under p of the generator of F_0 corresponding to the first row of φ .

Remark 1.2. If R has a symmetric minimal free resolution of length 2, i.e. a resolution of the form

$$0 \longrightarrow (A^n)^\vee \xrightarrow{\psi = \begin{pmatrix} -\beta^\vee \\ \alpha^\vee \end{pmatrix}} A^n \oplus (A^n)^\vee \xrightarrow{\varphi = (\alpha \ \beta)} A^n \xrightarrow{p} R \longrightarrow 0,$$

then the condition $\text{depth}(\text{Ann}_A R, A) = 2$ follows from the Eisenbud-Buchsbaum acyclicity criterion (cf. [Ei], thm. 20.9); for then the zeroth Fitting ideals of ψ and φ agree and we must have $\text{depth } I \geq 2$ for this complex to be exact. But $\text{rad } I = \text{rad } \text{Ann}_A R$ whence, in view of $\text{depth}(\text{Ann}_A R, A) \leq \text{projdim}_A R$, $\text{depth}(\text{Ann}_A R, A) = 2$.

Proof. The assumption that A be Cohen-Macaulay implies that R is a Cohen-Macaulay A -module, too; the argument is that the Auslander-Buchsbaum formula $\text{projdim } R + \text{depth } R = \text{depth } A = \dim A$ (the latter because A is Cohen-Macaulay) gives $\text{depth } R = \dim A - 2$, whereas $2 = \text{depth}(\text{Ann}_A R, A) = \dim A - \dim_A R$ (where again enters the hypothesis that A is Cohen-Macaulay whence the depth of an ideal is given by its codimension), and thus $\text{depth } R = \dim_A R$, i.e. R is Cohen-Macaulay.

It will be convenient to introduce the following notation; let

$$\left| \begin{array}{l} X := \text{Spec } A, \\ Y := \text{the codimension 2 closed subscheme of } X \text{ associated to } \text{Ann}_A R, \\ Z \subset Y \subset X \text{ the codimension 2 (in } Y) \text{ closed subscheme defined by } I', \\ U := Y - Z \text{ the open complement, } j : U \hookrightarrow Y \text{ the inclusion,} \\ \mathcal{R} \text{ resp. } \mathcal{C} \text{ the sheaves associated to } R \text{ resp. } C \text{ on } Y. \end{array} \right.$$

Now $\mathcal{R}|_U \cong \mathcal{O}_Y|_U$ (since the localization of the presentation matrix φ' for R/A_Y has invertible maximal minors), which is what I paraphrased in the sentence preceding theorem 1.1 saying that R is a ring "in codimension 1". One has the exact sequence relating local and global cohomologies (cf. [Groth], prop. 2.2)

$$0 \longrightarrow H_Z^0(Y, \mathcal{R}) \longrightarrow H^0(Y, \mathcal{R}) \longrightarrow H^0(U, \mathcal{R}) \longrightarrow H_Z^1(Y, \mathcal{R}) \longrightarrow 0;$$

On the other hand, since R is Cohen-Macaulay, $\text{depth}(I', R) = \dim R - \dim R/I'R \geq 2$, whence (cf. [Groth], thm. 3.8) $H_Z^0(Y, \mathcal{R}) = H_Z^1(Y, \mathcal{R}) = 0$ and thus $R \cong \Gamma(U, \mathcal{O}_Y|_U)$ (as A_Y -modules). Thus, via this isomorphism, we already know that R carries a structure of A_Y -(or if you like A -)algebra.

To prove that $R \cong \text{Hom}_{A_Y}(C, C)$ I first have to digress on some general points concerning the structure of R and C (cf. [E-U]): Let $e \in R$ denote the image under p of the A -module generator of F_0 corresponding to the first row of φ ; then φ' is a presentation matrix (over A) of $R/A_Y e$ whence

$I' \subset \text{Ann}_A R/A_Y e$ or $(I' \cdot A_Y)R \subset A_Y e$. But we have already seen that $\text{depth}(I', R) \geq 2$ and thus there is an element $d \in (I' \cdot A_Y) \subset A_Y$ which is a nonzerodivisor on R (therefore also on A_Y) with $dR \subset A_Y e \subset R$. That is, R is what is called a *finite birational* A -module in [E-U]. Incidentally, this implies that the algebra structure on R is unique since it is a subalgebra of $R[d^{-1}] = A[d^{-1}]$. This being said, the conductor $\text{Hom}_{A_Y}(R, A_Y)$ naturally identifies with the ideal $\text{Ann}_{A_Y}(R/A_Y e)$ in A_Y via the prescription that $a \in \text{Ann}_{A_Y}(R/A_Y e)$ should be identified with multiplication by a on R (to get a map back, send $c \in \text{Hom}_{A_Y}(R, A_Y)$ to $c(d)/d$). In fact, the conductor is also an ideal in R .

This last remark allows one to immediately conclude that $R^{**} \equiv \text{Hom}_{A_Y}(C, A_Y) = \text{Hom}_{A_Y}(C, C)$, cf. [Cat4], lemma 5.3 (here $*$ denotes the dualizing functor with respect to A_Y): Indeed, $\text{Hom}_{A_Y}(C, C) \subset \text{Hom}_{A_Y}(C, A_Y)$ being clear, let $\xi \in \text{Hom}_{A_Y}(C, A_Y)$ be given; let c be in C arbitrary. We have to show that $\xi c \in C$. But if r is in R , then cr is in C , since C is an ideal in R , thus ξcr is in A_Y and hence $\xi c \in C$.

We are now ready to conclude that $R \cong \text{Hom}_{A_Y}(C, C)$. Indeed, we know already that $\mathcal{R}|_U \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}, \mathcal{C})|_U$, since also $\mathcal{C}|_U \cong \mathcal{O}_Y|_U$. On the other hand, we know $\mathcal{R} \cong j_* j^* \mathcal{R}$ (from the exact sequence

$$0 \longrightarrow \mathcal{H}_Z^0(\mathcal{R}) \longrightarrow \mathcal{R} \longrightarrow j_* j^* \mathcal{R} \longrightarrow \mathcal{H}_Z^1(\mathcal{R}) \longrightarrow 0$$

(cf. [Groth], cor. 1.9) and $\text{depth}(I', R) \geq 2$). From the short exact sequence

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y/\mathcal{C} \longrightarrow 0$$

we get the long exact sequence

$$0 \longrightarrow \mathcal{H}_Z^0(\mathcal{C}) \longrightarrow \mathcal{H}_Z^0(\mathcal{O}_Y) \longrightarrow \mathcal{H}_Z^0(\mathcal{O}_Y/\mathcal{C}) \longrightarrow \mathcal{H}_Z^1(\mathcal{C}) \longrightarrow \dots$$

and $\mathcal{H}_Z^0(\mathcal{O}_Y) = 0$ because e.g. d above is a nonzerodivisor on A_Y , that is $\text{depth}(I', A_Y) \geq 1$. Thus also $\mathcal{H}_Z^0(\mathcal{C}) = 0$, which effectively means that the natural map $\mathcal{C} \longrightarrow j_* j^* \mathcal{C}$ is injective. Hence also the map $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}, \mathcal{C}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}, j_* j^* \mathcal{C}) \cong j_* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}, \mathcal{C})|_U$ is an injection. Summing up, we can build a commutative diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\iota_1} & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}, \mathcal{O}_Y) \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}, \mathcal{C}) \\ \downarrow \varepsilon_1 & & \downarrow \iota_2 \\ j_* \mathcal{R}|_U & \xrightarrow{\varepsilon_2} & j_* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}, \mathcal{C})|_U \end{array}$$

where ε_1 and ε_2 are isomorphisms and ι_1, ι_2 injections (ι_1 is just the natural inclusion of \mathcal{R} into its bidual). Thus ι_1 and ι_2 are likewise isomorphisms

whence $\mathcal{R} \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}, \mathcal{C})$ and by taking global sections the desired conclusion $R \cong \text{Hom}_{A_Y}(C, C)$. \blacksquare

Thus, if one puts the condition (\heartsuit) , the fact that R is a ring falls in one's lap almost automatically, a philosophy already supported in [Cat4] (see the discussion following remark 6.6); but the situation now appears to be even more idyllic than it was hinted at to be in the latter reference. Note that the crucial property working behind the scenes and making all the above extension arguments work is only the Cohen-Macaulayness of R .

Theorem 1.1 has the following analogue in the realm of graded rings. (In the statement of the result I will slightly abuse notation falling back on the previously encountered symbols R etc.; but this is probably more transparent than introducing too many new letters).

Theorem 1.3 (theorem 1.1 bis). *Let $S = S_0 \oplus S_1 \oplus \dots$ be a graded Cohen-Macaulay ring such that S_0 is a field and S a finitely generated S_0 -algebra. Let R be a (graded) finite S -module with a minimal graded free resolution*

$$0 \longrightarrow \bigoplus_l S(-s_l) \xrightarrow{\psi} \bigoplus_k S(-r_k) \xrightarrow{\varphi} \bigoplus_j S(-q_j) \xrightarrow{p} R \longrightarrow 0.$$

Let φ' be the submatrix of φ consisting of all the rows of φ except the first and denote by I resp. I' the zeroth Fitting ideals of φ resp. φ' . Suppose that $\text{depth}(\text{Ann}_S R, S) = 2$ and

$$(\heartsuit) \text{ depth } I' \geq 4.$$

Then $R \cong \text{Hom}_{S_Y}(C, C)$, where $S_Y := S/\text{Ann}_S R$ and $C := \text{Hom}_{S_Y}(R, S_Y)$, the conductor of R into S_Y ; R is given an S -algebra structure if we define multiplication as composition of endomorphisms of C .

The proof is completely analogous to that of theorem 1.1. The assumptions on S are made to be able to apply the Auslander-Buchsbaum formula in the graded case to conclude as before that R is Cohen-Macaulay. One then works with the sheaf \mathcal{R} associated to R on $\text{Spec } S_Y$, the affine cone over $\text{Proj } S_Y$, to finish the argument.

2 The Gorenstein condition and self-duality of the Hilbert resolution

In this section I tie theorem 1.1 in with the symmetry properties that are native to Gorenstein algebras; in fact, this aspect of the problem has already

received satisfactory treatment in [Gra], but I will outline proofs for the sake of completeness (also because the arguments in [Gra] are closely interwoven with the somewhat independent concept of a *Koszul module* and occasionally there is room for simplifications).

The basic set-up will be kept. In particular A is a Cohen-Macaulay local ring and R a finite A -module. First recall

Definition 2.1. If R is a perfect A -algebra (meaning that $\text{depth}(\text{Ann}_A R, A) = \text{projdim}_A R$) and $R \cong \text{Ext}_A^c(R, A)$ as R -modules (where $c = \dim A - \dim_A R$) then R is said to be a *Gorenstein A -algebra of codimension c* .

Remark 2.2. The R -module structure on the a priori A -module $\text{Ext}_A^c(R, A)$ is induced from R by functoriality of $\text{Ext}_A^c(\cdot, A)$: Thus if for $r \in R$ mult_r is multiplication by r on R , then $\text{Ext}_A^c(\text{mult}_r, A)$ is multiplication by r on $\text{Ext}_A^c(R, A)$.

Similarly if $S = S_0 \oplus S_1 \oplus \dots$ is a positively graded Cohen-Macaulay ring with S_0 a field, and S finitely generated over S_0 as an algebra, one can make

Definition 2.3. A finite perfect graded S -algebra B is called a *Gorenstein S -algebra of codimension c* (and *with twist $t \in \mathbb{Z}$*) if $B \cong \text{Ext}_S^c(B, S(t))$ as B -modules where $c = \dim S - \dim_S B$.

Then we have the following characterization:

Theorem 2.4. *If the finite A -module R (A CM local and 2 invertible in A) is a Gorenstein algebra of codimension 2, then it admits a symmetric minimal free resolution of the form*

$$\mathbf{R}_\bullet : 0 \longrightarrow (A^n)^\vee \xrightarrow{\psi = \begin{pmatrix} -\beta^\vee \\ \alpha^\vee \end{pmatrix}} A^n \oplus (A^n)^\vee \xrightarrow{\varphi = (\alpha \beta)} A^n \xrightarrow{p} R \longrightarrow 0.$$

Conversely, if I' denotes the zeroth Fitting ideal of the submatrix φ' of φ consisting of all the rows of φ except the first and if we have $\text{depth } I' \geq 4$, then the existence of a symmetric minimal free resolution of the form \mathbf{R}_\bullet is also sufficient for the A -module R to be a Gorenstein A -algebra of codimension 2.

Proof. Let me start with proving the converse: Thus one is given a finite A -module R with resolution \mathbf{R}_\bullet and $\text{depth } I' \geq 4$; by theorem 1.1 and remark 1.2 one knows that R carries the structure of a commutative A -algebra with 1, and moreover $\text{depth}(\text{Ann}_A R, A) = \min\{i : \text{Ext}_A^i(R, A) \neq 0\} = 2$ whence I can dualize \mathbf{R}_\bullet and build the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\begin{pmatrix} -\beta^\vee \\ \alpha^\vee \end{pmatrix}} & A^n \oplus (A^n)^\vee & \xrightarrow{(\alpha \ \beta)} & A^n & \longrightarrow & R & \longrightarrow & 0 \\
& & \downarrow \text{id}_{(A^n)^\vee} & & \downarrow \begin{pmatrix} 0 & \text{id}_{(A^n)^\vee} \\ -\text{id}_{A^n} & 0 \end{pmatrix} & & \downarrow \text{id}_{A^n} & & \downarrow u \cong & & \\
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\begin{pmatrix} \alpha^\vee \\ \beta^\vee \end{pmatrix}} & (A^n)^\vee \oplus A^n & \xrightarrow{(-\beta \ \alpha)} & A^n & \longrightarrow & \text{Ext}_A^2(R, A) & \longrightarrow & 0
\end{array}$$

where $u : R \cong \text{Ext}_A^2(R, A)$ is the A -module isomorphism induced by the diagram. I claim that this is also an isomorphism of R -modules: indeed, from the proof of theorem 1.1 we know that R is a subring of $A_Y[d^{-1}]$ where $A_Y := A/\text{Ann}_A R$ and $d \in A_Y$ is some nonzerodivisor. Let then a/d , $a \in A_Y$, be some element of R . Then for $r \in R$ we have $u(d(a/d)r) = au(r) = du((a/d)r)$, i.e. $u((a/d)r) = (a/d)u(r)$ and u is also R -linear.

To prove the other direction, let R be a Gorenstein algebra of codimension 2, and let an isomorphism $u : R \cong \text{Ext}_A^2(R, A)$ be given. Since R is perfect and $\dim A - \dim_A R = \text{depth}(\text{Ann}_A R, A) = 2$ (A is CM) we can take a minimal free length 2 resolution of R

$$0 \longrightarrow F_2 \xrightarrow{\Psi} F_1 \xrightarrow{\Phi} F_0 \longrightarrow R \longrightarrow 0$$

where from $R \cong \text{Ext}_A^2(R, A)$ it follows that $\text{rank } F_0 = \text{rank } F_2$, and since $\text{Ann}_A R \neq 0$ $\text{rank } F_0 - \text{rank } F_1 + \text{rank } F_2 = 0$ whence I can write $F_0 \cong A^n$, $F_1 \cong A^n \oplus (A^n)^\vee$, $F_2 \cong (A^n)^\vee$ for some integer n . Dualize this resolution to obtain a minimal free resolution of $\text{Ext}_A^2(R, A)$ and lift the given isomorphism u to an isomorphism of minimal free resolutions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\Psi} & A^n \oplus (A^n)^\vee & \xrightarrow{\Phi} & A^n & \longrightarrow & R & \longrightarrow & 0 \\
& & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow u \cong & & \\
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\Phi^\vee} & (A^n)^\vee \oplus A^n & \xrightarrow{\Psi^\vee} & A^n & \longrightarrow & \text{Ext}_A^2(R, A) & \longrightarrow & 0
\end{array}$$

(Here and in the following I always implicitly identify free modules and maps between them with their double duals). Dualizing once more, we can build the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\Psi} & A^n \oplus (A^n)^\vee & \xrightarrow{\Phi} & A^n & \longrightarrow & R & \longrightarrow & 0 \\
& & \downarrow \text{id}_{(A^n)^\vee} & & \downarrow \text{id}_{A^n \oplus (A^n)^\vee} & & \downarrow \text{id}_{A^n} & & \downarrow c \cong & & \\
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\Psi} & A^n \oplus (A^n)^\vee & \xrightarrow{\Phi} & A^n & \longrightarrow & \text{Ext}_A^2(\text{Ext}_A^2(R, A), A) & \longrightarrow & 0 \\
& & \downarrow f_1^\vee & & \downarrow f_2^\vee & & \downarrow f_3^\vee & & \downarrow \text{Ext}_A^2(u, A) & & \\
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\Phi^\vee} & (A^n)^\vee \oplus A^n & \xrightarrow{\Psi^\vee} & A^n & \longrightarrow & \text{Ext}_A^2(R, A) & \longrightarrow & 0
\end{array}$$

where c is the canonical isomorphism determined by the diagram. Now the point that needs some work is to compare the isomorphisms u and $\text{Ext}_A^2(u, A) \circ c$. This is done in [Gra] (cf. especially lemma 2.1 and thm. 3) by identifying the functors $\text{Ext}_A^2(\cdot, A)$ resp. $\text{Ext}_A^2(\text{Ext}_A^2(\cdot, A), A)$ with $\text{Hom}_A(\cdot, A/(x_1, x_2))$ resp. $\text{Hom}_A(\text{Hom}_A(\cdot, A/(x_1, x_2)), A/(x_1, x_2))$ where x_1, x_2 is a regular sequence in $\text{Ann}_A R$, and making all of the occurring isomorphisms explicit. Grassi uses the extra assumption that A be a domain to make the proof work, but this is in fact redundant as I will show in lemma 3.2 below (the argument needed to remove this hypothesis is a little technical, so I deferred it to the next section where, after all, it integrates rather better). It turns out that $\text{Ext}_A^2(u, A) \circ c = -u$, and since 2 was supposed to be invertible in A , one can conclude that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\Psi} & A^n \oplus (A^n)^\vee & \xrightarrow{\Phi} & A^n & \longrightarrow & R & \longrightarrow & 0 \\
& & \downarrow \frac{f_3 - f_1^\vee}{2} & & \downarrow \frac{f_2 - f_2^\vee}{2} & & \downarrow \frac{f_1 - f_3^\vee}{2} & & \downarrow u \cong & & \\
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\Phi^\vee} & (A^n)^\vee \oplus A^n & \xrightarrow{\Psi^\vee} & A^n & \longrightarrow & \text{Ext}_A^2(R, A) & \longrightarrow & 0
\end{array}$$

Then $(f_2 - f_2^\vee)/2$ is a skew isomorphism which, by a suitable orthogonal isomorphism B of $A^n \oplus (A^n)^\vee$, $B^\vee B = \text{id}_{A^n \oplus (A^n)^\vee}$, can be brought to normal form

$$J := \begin{pmatrix} 0 & \text{id}_{(A^n)^\vee} \\ -\text{id}_{A^n} & 0 \end{pmatrix} = B^\vee \begin{pmatrix} f_2 - f_2^\vee \\ 2 \end{pmatrix} B.$$

One is done because the lower row of the following commutative diagram is

of the form required in resolution \mathbf{R}_\bullet :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{\Psi} & A^n \oplus (A^n)^\vee & \xrightarrow{\Phi} & A^n & \longrightarrow & R & \longrightarrow & 0 \\
& & \downarrow \frac{f_3 - f_1^\vee}{2} & & \downarrow B^\vee & & \downarrow \text{id}_{A^n} & & \downarrow \text{id}_R & & \\
0 & \longrightarrow & (A^n)^\vee & \xrightarrow{-J(\Phi \circ B)^\vee} & A^n \oplus (A^n)^\vee & \xrightarrow{\Phi \circ B} & A^n & \longrightarrow & \text{Ext}_A^2(R, A) & \longrightarrow & 0
\end{array}$$

■

Finally let me state the theorem corresponding to theorem 2.4 in the graded case whose proof is entirely similar to the preceding one.

Theorem 2.5 (theorem 2.4 bis). *Let $S = S_0 \oplus S_1 \oplus \dots$ be a graded Cohen-Macaulay ring such that S_0 is a field and S a finitely generated S_0 -algebra. Assume $\text{char } S_0 \neq 2$. Then every Gorenstein S -algebra of codimension 2 and with twist t has a symmetric graded free resolution*

$$\begin{aligned}
\mathbf{R}_\bullet : 0 \longrightarrow \bigoplus_{j=1}^n S^\vee(-t + r_j) &\xrightarrow{\psi = \begin{pmatrix} -\beta^\vee \\ \alpha^\vee \end{pmatrix}} \bigoplus_{k=1}^n S^\vee(-t + s_k) \oplus \bigoplus_{k=1}^n S(-s_k) \\
&\xrightarrow{\varphi = (\alpha \beta)} \bigoplus_{j=1}^n S(-r_j) \xrightarrow{p} R \longrightarrow 0,
\end{aligned}$$

$(n \in \mathbb{N}, t, r_j, s_k \in \mathbb{Z})$. Conversely, if I' denotes the zeroth Fitting ideal of the matrix φ' which is the matrix φ with first row erased and $\text{depth } I' \geq 4$, then a graded S -module R with a symmetric minimal graded free resolution of the form \mathbf{R}_\bullet is a Gorenstein S -algebra of codimension 2 and with twist t .

■

The proof is almost entirely similar to the preceding one, but there is one point that deserves mentioning: When proving that the Gorenstein S -algebra R has a symmetric graded free resolution, I want to use that the given isomorphism $R \cong \text{Ext}_S^2(R, S(t))$ is skew-symmetric with respect to the duality given by $\text{Ext}_S^2(-, S(t))$. In the local case this followed from [Gra]. But then, in particular, in the present situation, one knows that for every $\mathfrak{p} \in \text{Supp } R$ the localized isomorphism $R_{\mathfrak{p}} \cong \text{Ext}_{S_{\mathfrak{p}}}^2(R_{\mathfrak{p}}, S_{\mathfrak{p}})$ is skew-symmetric, whence also the original isomorphism $R \cong \text{Ext}_S^2(R, S(t))$ is skew-symmetric.

3 Regularizing minors

This section provides a strengthening of theorem 2.4, the local case of the structure theorem, and thereby clarifies the relationship between Gorenstein algebras in codimension 2 and Koszul modules as introduced in [Gra]. Therefore let (A, \mathfrak{m}, k) be again CM local with $2 \notin \mathfrak{m}$ and R a codimension 2 Gorenstein algebra over A . Whereas the usual Koszul complex is associated with a linear form $f : A^n \rightarrow A$, a Koszul module is a module having a resolution similar to the Koszul complex up to the fact that the rôle of f is taken by a family of (vector-valued) maps from A^n to A^n . I'll only make this precise in the relevant special case:

Definition 3.1. A finite A -module M having a length 2 resolution

$$0 \rightarrow A^n \xrightarrow{\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}} A^{2n} \xrightarrow{(\tau_1 \ \tau_2)} A^n \rightarrow M \rightarrow 0 \quad (1)$$

some $n \in \mathbb{N}$, is a Koszul module iff $\det(\tau_1), \det(\tau_2)$ is a regular sequence on A and \exists a unit $\lambda \in A$: $\det(\rho_1) = (-1)^n \lambda \det(\tau_2)$, $\det(\rho_2) = \lambda \det(\tau_1)$.

Then Grassi proves in case A is a domain ([Gra], thm. 3.3) that R has a (Gorenstein) symmetric resolution

$$0 \rightarrow A^n \xrightarrow{\begin{pmatrix} -\beta^t \\ \alpha^t \end{pmatrix}} A^{2n} \xrightarrow{(\alpha \ \beta)} A^n \rightarrow R \rightarrow 0 \quad (2)$$

and a second resolution of the prescribed type (1) for the Koszul module condition, and that these 2 are related by an isomorphism of complexes which is the identity in degrees 0 and 2; firstly, for sake of generality, I will briefly show that the assumption " A a domain" is in fact not needed, thereby closing also the remaining gap in the proof of theorem 2.4, and secondly, prove that there is one single resolution of R meeting both requirements, i.e. a resolution as in (2) with $\det(\alpha), \det(\beta)$ an A -regular sequence.

Lemma 3.2. *A Gorenstein algebra R has a minimal free resolution of type (2) over any CM local ring with $2 \notin \mathfrak{m}$ (i.e. one need not assume that A be a domain).*

Proof. Note that the only place in [Gra] where the hypothesis that A be a domain enters is at the beginning of the proof of proposition 1.5, page 930: Here one is given a resolution as in (1), but without any additional assumptions on $\det(\tau_1), \det(\tau_2), \det(\rho_1), \det(\rho_2)$ whatsoever, and Grassi wants to conclude that \exists a base change in A^{2n} such that (in the new base) $\det(\tau_1)$ is not a zero divisor on A . But this can be proven by a similar method as

Grassi uses in the sequel of the proof of proposition 1.5, without using "A a domain": For let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the associated primes of A which are precisely the minimal elements of $\text{Spec}(A)$ since A is CM. One shows that \exists a base change in A^{2n} such that $\det(\tau_1) \notin \mathfrak{p}_i, \forall i = 1, \dots, r$ (in the new base), more precisely, that \exists a sequence of r base changes such that after the m th base change

$$(*) \quad \det(\tau_1) \notin \mathfrak{p}_i, \forall i \in \{r - m + 1, \dots, r\},$$

$m = 0, \dots, r$, the assertion being empty for $m = 0$. Therefore, inductively, suppose $(*)$ holds for m to get it for $m + 1$.

Denote by $[i_1, \dots, i_n]$ the maximal minor of $(\tau_1 \ \tau_2)$ corresponding to the columns $i_1, \dots, i_n, i_j \in \{1, \dots, 2n\}$. If $[1, \dots, n] \notin \mathfrak{p}_{r-m}$ I'm already O.K., so suppose $[1, \dots, n] \in \mathfrak{p}_{r-m}$. By the Eisenbud-Buchsbaum acyclicity criterion $I_n((\tau_1 \ \tau_2))$ cannot consist of zerodivisors on A alone, therefore set

$$l_1 := \min\{c : \exists s_1, \dots, s_{n-1} \text{ with } s_1 < s_2 < \dots < s_{n-1} < c \\ \text{and } [s_1, \dots, s_{n-1}, c] \notin \mathfrak{p}_{r-m}\}$$

(then $n < l_1 \leq 2n$) and inductively,

$$l_i := \min\{c : \exists s'_1, \dots, s'_{n-i} \text{ with } s'_1 < \dots < s'_{n-i} < c < l_{i-1} < \dots < l_1 \\ \text{and } [s'_1, \dots, s'_{n-i}, c, l_{i-1}, \dots, l_1] \notin \mathfrak{p}_{r-m}\},$$

$i = 2, \dots, n$. Then $\exists J$ such that $n < l_J < l_{J-1} < \dots < l_1 \leq 2n$ and for $I > J$ $l_I \in \{1, \dots, n\}$ ($J = n$ might occur and then the set of $l_I \in \{1, \dots, n\}$ is empty; this does not matter).

I have $[l_n, \dots, l_1] \notin \mathfrak{p}_{r-m}$ by construction. Choose $b \in (\bigcap_{i=r-m+1}^r \mathfrak{p}_i) \setminus \mathfrak{p}_{r-m}$,

which is nonempty since the \mathfrak{p}_i 's are the minimal elements of $\text{Spec}(A)$. Denote by $y_1 < \dots < y_J$ the complementary indices of the $l_I \in \{1, \dots, n\}$ inside $\{1, \dots, n\}$ and consider the base change on A^{2n} : $M_{y_1, l_J}(b) \circ M_{y_2, l_{J-1}}(b) \circ \dots \circ M_{y_J, l_1}(b)$, where $M_{y_\nu, l_{J-\nu+1}}(b)$, $\nu = 1, \dots, J$ is addition of b times the $l_{J-\nu+1}$ column to the y_ν column. Then one sees (by the multilinearity of determinants)

$$[1, \dots, n]_{\text{new}} = [1, \dots, n]_{\text{old}} \pm b^J [l_n, \dots, l_1]_{\text{old}} + b\mu,$$

where "new" means after and "old" before the base change and μ is an element in \mathfrak{p}_{r-m} by the defining minimality property of the l 's. Therefore, since by the induction hypothesis $[1, \dots, n]_{\text{old}} \notin \mathfrak{p}_i, \forall i \in \{r - m + 1, \dots, r\}$ and b is chosen appropriately: $[1, \dots, n]_{\text{new}} \notin \mathfrak{p}_i, \forall i \in \{r - m, \dots, r\}$. This finally proves $\det(\tau_1) \notin \mathfrak{p}_i \forall i = 1, \dots, r$ after the sequence of base changes, i.e. $\det(\tau_1)$ is then A -regular, that what was to be shown. \blacksquare

Secondly, I now want to prove:

Theorem 3.3. *A codimension 2 Gorenstein algebra R over a local CM ring (A, \mathfrak{m}, k) with $2 \notin \mathfrak{m}$ has a resolution*

$$0 \rightarrow A^n \xrightarrow{\begin{pmatrix} -\beta^t \\ \alpha^t \end{pmatrix}} A^{2n} \xrightarrow{(\alpha \ \beta)} A^n \rightarrow R \rightarrow 0$$

which is also of Koszul module type, i.e. $\det(\alpha), \det(\beta)$ is an A -regular sequence.

Proof. Taking into account the above remark that one can dispose of the assumption "A a domain" the fact that R has a resolution with the symmetry property above is proven in [Gra], thm. 3.3., so I have to show that \exists a base change in A^{2n} which preserves the relation $\alpha\beta^t = \beta\alpha^t$ and in the new base $\det(\alpha), \det(\beta)$ is a regular sequence. The punch line to show this is as in the foregoing argument except that everything is a little harder because one has to keep track of preserving the symmetry: Therefore let again be $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ the associated primes of A , and I show that \exists a sequence of r base changes in A^{2n} preserving the symmetry and such that after the m th base change $(*)$ above holds, the case $m = 0$ being trivial. For the inductive step, suppose $\det(\alpha) \in \mathfrak{p}_{r-m}$ to rule out a trivial case; I write $[i_1, \dots, i_\nu; j_1, \dots, j_{n-\nu}] \equiv \det(\alpha_{i_1} \dots \alpha_{i_\nu} \beta_{j_1} \dots \beta_{j_{n-\nu}})$. Call a minor $[i_1, \dots, i_\nu; j_1, \dots, j_{n-\nu}]$ good iff $\{i_1, \dots, i_\nu\} \cap \{j_1, \dots, j_{n-\nu}\} = \emptyset$.

I want to find a good minor that does not belong to \mathfrak{p}_{r-m} (possibly after a base change in A^{2n}). Therefore suppose all the good minors belong to \mathfrak{p}_{r-m} . Since $\text{grade } I_n((\alpha \ \beta)) \geq 2$ by Eisenbud-Buchsbaum acyclicity, \exists a minor $\notin \mathfrak{p}_{r-m}$ (which is not good). For $n = 1$ this is a contradiction since all minors are good, and I can suppose $n > 1$ in the process of finding a good minor. Now choose a minor $[I_1, \dots, I_k; J_1, \dots, J_{n-k}]$ such that

- $[I_1, \dots, I_k; J_1, \dots, J_{n-k}] \notin \mathfrak{p}_{r-m}$
- $\text{card}(\{I_1, \dots, I_k\} \cap \{J_1, \dots, J_{n-k}\}) =: M_0$ is minimal among the minors which do not belong to \mathfrak{p}_{r-m} .

I want to perform a base change in A^{2n} not destroying the symmetry such that in the new base \exists a minor $[T_1, \dots, T_{k-1}; S_1, \dots, S_{n-k+1}]$ such that

- $[T_1, \dots, T_{k-1}; S_1, \dots, S_{n-k+1}] \notin \mathfrak{p}_{r-m}$
- $\text{card}(\{T_1, \dots, T_{k-1}\} \cap \{S_1, \dots, S_{n-k+1}\}) = M_0 - 1$.

Continuing this process M_0 steps (i.e. performing M_0 successive base changes) I can find a good minor not contained in \mathfrak{p}_{r-m} .

Let now $[T_1, \dots, T_{k-1}; S_1, \dots, S_{n-k+1}]$ be given. Choose $H \in \{I_1, \dots, I_k\} \cap \{J_1, \dots, J_{n-k}\}$ and $L \in \{1, \dots, n\} - \{I_1, \dots, I_k\} \cup \{J_1, \dots, J_{n-k}\}$ (both of which exist). Now perform the base change in A^{2n} which corresponds to adding α_H to β_L and α_L to β_H (preserving the symmetry), and consider

$$\det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \beta_{J_1} \dots \beta_H + \alpha_L \dots \beta_L + \alpha_H \dots \beta_{J_{n-k}}),$$

an $n \times n$ -minor of the transformed matrix which I can write as $[T_1, \dots, T_{k-1}; S_1, \dots, S_{n-k+1}]$, where $\{T_1, \dots, T_{k-1}\} = \{I_1, \dots, I_k\} - \{H\}$, $\{S_1, \dots, S_{n-k+1}\} = \{J_1, \dots, J_{n-k}\} \cup \{L\}$ and obviously, $\text{card}(\{T_1, \dots, T_{k-1}\} \cap \{S_1, \dots, S_{n-k+1}\}) = M_0 - 1$. I want to prove that this minor does not belong to \mathfrak{p}_{r-m} . For this I show that in fact

$$[T_1, \dots, T_{k-1}; S_1, \dots, S_{n-k+1}] = \pm [I_1, \dots, I_k; J_1, \dots, J_{n-k}] + \text{"residual terms"},$$

where "residual terms" $\in \mathfrak{p}_{r-m}$. Using the additivity of the determinant in each column I find that "residual terms" consists of 3 summands two of which clearly belong to \mathfrak{p}_{r-m} because $[I_1, \dots, I_k; J_1, \dots, J_{n-k}]$ was chosen such that $\text{card}(\{I_1, \dots, I_k\} \cap \{J_1, \dots, J_{n-k}\}) =: M_0$ was minimal among the minors of the matrix before the base change which did not belong to \mathfrak{p}_{r-m} , whereas the third summand is (up to sign)

$$\det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \alpha_L \beta_{J_1} \dots \hat{\beta}_H \dots \beta_{J_{n-k}} \beta_L).$$

To show that the latter is in \mathfrak{p}_{r-m} I apply the so-called "Plücker relations":

Given an $M \times N$ -matrix, $M \leq N$, $a_1, \dots, a_p, b_q, \dots, b_M, c_1, \dots, c_s \in \{1, \dots, N\}$, $s = M - p + q - 1 > M$, $t = M - p > 0$, one has

$$(P) \quad \sum_{\substack{i_1 < \dots < i_t \\ i_{t+1} < \dots < i_s \\ \{i_1, \dots, i_s\} = \{1, \dots, s\}}} \sigma(i_1, \dots, i_s) [a_1, \dots, a_p c_{i_1} \dots c_{i_t}] [c_{i_{t+1}} \dots c_{i_s} b_q \dots b_M] = 0$$

where $\sigma(i_1, \dots, i_s)$ is the sign of the permutation $\binom{1, \dots, s}{i_1, \dots, i_s}$ (see e.g. [B-He], lemma 7.2.3, p. 308).

In my situation, I let $M := n$, $N := 2n$, $p := n - 2$, $q := n + 1$, $s := n + 1$ and for the columns corresponding to the a 's above I choose the $n - 2$ columns

$$\alpha_{I_1}, \alpha_{I_2}, \dots, \hat{\alpha}_H, \dots, \alpha_{I_k}, \beta_{J_1}, \dots, \hat{\beta}_H, \dots, \beta_{J_{n-k}}$$

(in this order), for the columns corresponding to the b 's I choose the empty set (which is allowable here), and finally for the columns corresponding to the c 's the $n + 2$ columns

$$\alpha_H, \beta_H, \alpha_L, \beta_L, \alpha_{I_1}, \alpha_{I_2}, \dots, \hat{\alpha}_H, \dots, \alpha_{I_k}, \beta_{J_1}, \dots, \hat{\beta}_H, \dots, \beta_{J_{n-k}}$$

Applying (P) one gets 6 nonvanishing summands, 4 of which (namely

$$\begin{aligned} & \det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \beta_{J_1} \dots \hat{\beta}_H \dots \beta_{J_{n-k}} \alpha_H \alpha_L) \cdot (\text{a second factor}), \\ & \det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \beta_{J_1} \dots \hat{\beta}_H \dots \beta_{J_{n-k}} \alpha_H \beta_L) \cdot (\text{a second factor}), \\ & \det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \beta_{J_1} \dots \hat{\beta}_H \dots \beta_{J_{n-k}} \beta_H \alpha_L) \cdot (\text{a second factor}), \\ & \det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \beta_{J_1} \dots \hat{\beta}_H \dots \beta_{J_{n-k}} \beta_H \beta_L) \cdot (\text{a second factor}) \end{aligned}$$

are in \mathfrak{p}_{r-m} by the defining minimality property of $[I_1, \dots, I_k; J_1, \dots, J_{n-k}]$ above. The remaining 2 summands add up to (watch the signs!)

$\pm 2 \det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \alpha_L \beta_{J_1} \dots \hat{\beta}_H \dots \beta_{J_{n-k}} \beta_L) \cdot [I_1, \dots, I_k; J_1, \dots, J_{n-k}]$ which therefore is also in \mathfrak{p}_{r-m} . But $[I_1, \dots, I_k; J_1, \dots, J_{n-k}] \notin \mathfrak{p}_{r-m}$ and 2 is a unit in A , therefore $\det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \alpha_L \beta_{J_1} \dots \hat{\beta}_H \dots \beta_{J_{n-k}} \beta_L) \in \mathfrak{p}_{r-m}$ as desired, since \mathfrak{p}_{r-m} is prime.

Hence inductively, after M_0 base changes in A^{2n} , I can find a good minor of the transformed matrix that is not in \mathfrak{p}_{r-m} . I assume $[1, \dots, n] \in \mathfrak{p}_{r-m}$. I can now define

$$\begin{aligned} l_1 &:= \min\{c : \exists s_1, \dots, s_{n-1} \text{ with } s_1 < s_2 < \dots < s_{n-1} < c \\ &\quad \text{and } [s_1, \dots, s_{n-1}, c] \notin \mathfrak{p}_{r-m} \text{ and } [s_1, \dots, s_{n-1}, c] \text{ is good}\} \end{aligned}$$

(then $n < l_1 \leq 2n$) and inductively,

$$\begin{aligned} l_i &:= \min\{c : \exists s'_1, \dots, s'_{n-i} \text{ with } s'_1 < \dots < s'_{n-i} < c < l_{i-1} < \dots < l_1 \\ &\quad \text{and } [s'_1, \dots, s'_{n-i}, c, l_{i-1}, \dots, l_1] \text{ is good} \\ &\quad \text{and } [s'_1, \dots, s'_{n-i}, c, l_{i-1}, \dots, l_1] \notin \mathfrak{p}_{r-m}\}. \end{aligned}$$

Then $[l_n, \dots, l_1] \notin \mathfrak{p}_{r-m}$ which is good and can therefore be written as $[l_n, \dots, l_1] = [l_1^\alpha, \dots, l_h^\alpha, l_1^\beta, \dots, l_{n-h}^\beta]$ with $\{l_1^\alpha, \dots, l_h^\alpha\} \cap \{l_1^\beta, \dots, l_{n-h}^\beta\} = \emptyset$.

Choose $b \in (\bigcap_{i=r-m+1}^r \mathfrak{p}_i) \setminus \mathfrak{p}_{r-m}$ and perform a base change in A^{2n} (preserving

the symmetry) by adding b times the l_i^β column of β to the l_i^β column of α , for $i = 1, \dots, n - h$. Then

$$[1, \dots, n]_{\text{new}} = [1, \dots, n]_{\text{old}} \pm b^{n-h} [l_n, \dots, l_1]_{\text{old}} + b\mu,$$

where $\mu \in \mathfrak{p}_{r-m}$ by the defining minimality property of the l 's. Thus $[1, \dots, n]_{\text{new}} \notin \mathfrak{p}_i$ for $i = r - m, \dots, r$, which is the inductive step for the property (*). Therefore after a sequence of base changes that preserve the symmetry $\alpha\beta^t = \beta\alpha^t$, $\det(\alpha)$ can be made an A -regular element.

Let's sum up: I have that $\det(\alpha)$ is a nonzerodivisor in A , and want to prove that \exists a base change in A^{2n} preserving the symmetry and leaving α unchanged (i.e. fixing the first n basis vectors of A^{2n}) such that in the new base $\det(\beta)$ is a nonzerodivisor in $A/(\det(\alpha))$. The argument is almost identical to the preceding one. In fact, let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the associated primes of $A/(\det(\alpha))$ which are exactly the minimal prime ideals containing $(\det(\alpha))$ because $A/(\det(\alpha))$ is CM (A is CM and $\det(\alpha)$ is A -regular). Then the part of the above proof starting with "... the symmetry: Therefore let again be $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ the associated primes of A , and I show that \exists a sequence of r base changes in A^{2n} ..." and ending with "... Choose $H \in \{I_1, \dots, I_k\} \cap \{J_1, \dots, J_{n-k}\}$ and $L \in \{1, \dots, n\} - \{I_1, \dots, I_k\} \cup \{J_1, \dots, J_{n-k}\}$..." goes through verbatim (and has to be inserted here) if throughout one replaces r with s , $\det(\alpha)$ with $\det(\beta)$, and the symbol " \mathfrak{p} " with " \mathfrak{q} ". Thereafter, a slight change is necessary because in the process of finding a good minor, i.e. in the course of the M_0 base changes on A^{2n} that transform $(\alpha \beta)$ s.t. in the new base \exists a good minor, the shape of β is changed. This change must preserve the property $\det(\beta) \notin \mathfrak{q}_1, \dots, \mathfrak{q}_{s-m+1}$ in order not to destroy the induction hypothesis. The way out is as follows:

Choose $\zeta \in (\bigcap_{i=r-m+1}^r \mathfrak{q}_i) \setminus \mathfrak{q}_{r-m}$, which is possible since the \mathfrak{q} 's all have height

1. Now perform the base change in A^{2n} which corresponds to adding $\zeta\alpha_H$ to β_L and $\zeta\alpha_L$ to β_H (preserving the symmetry), and consider

$$\det(\alpha_{I_1} \dots \alpha_{I_k} \beta_{J_1} \dots \beta_{J_{n-k}} + \zeta\alpha_L \dots \beta_L + \zeta\alpha_H \dots \beta_{J_{n-k}}),$$

an $n \times n$ -minor of the transformed matrix which I can write as $[T_1, \dots, T_{k-1}; S_1, \dots, S_{n-k+1}]$, where $\{T_1, \dots, T_{k-1}\} = \{I_1, \dots, I_k\} - \{H\}$, $\{S_1, \dots, S_{n-k+1}\} = \{J_1, \dots, J_{n-k}\} \cup \{L\}$ and obviously, $\text{card}(\{T_1, \dots, T_{k-1}\} \cap \{S_1, \dots, S_{n-k+1}\}) = M_0 - 1$. I want to prove that this minor does not belong to \mathfrak{q}_{s-m} and furthermore that

$$\det(\beta_1 \dots \beta_H + \zeta\alpha_L \dots \beta_L + \zeta\alpha_H \dots \beta_n) \notin \mathfrak{q}_1, \dots, \mathfrak{q}_{s-m+1}.$$

The latter statement is obvious by the choice of ζ (and multilinearity of determinants). The former one follows if I show

$$[T_1, \dots, T_{k-1}; S_1, \dots, S_{n-k+1}] = \pm \zeta [I_1, \dots, I_k; J_1, \dots, J_{n-k}] + \text{"residual terms"},$$

where "residual terms" $\in \mathfrak{q}_{s-m}$ because ζ and $[I_1, \dots, I_k; J_1, \dots, J_{n-k}]$ are both $\notin \mathfrak{q}_{s-m}$ by assumption. Again "residual terms" consists of 3 summands two of which belong to \mathfrak{q}_{s-m} because of the defining minimality property of $[I_1, \dots, I_k; J_1, \dots, J_{n-k}]$. The third summand is up to sign

$$\zeta \det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \alpha_L \beta_{J_1} \dots \hat{\beta}_H \dots \beta_{J_{n-k}} \beta_L),$$

therefore it suffices to show $\det(\alpha_{I_1} \dots \hat{\alpha}_H \dots \alpha_{I_k} \alpha_L \beta_{J_1} \dots \hat{\beta}_H \dots \beta_{J_{n-k}} \beta_L) \in \mathfrak{q}_{s-m}$. This is done word by word as in the passage of the first part of this proof starting with "... I apply the so-called "Plücker relations":..." and ending "... since \mathfrak{p}_{r-m} is prime...", taking into account the aforementioned changes in notation.

The rest of the proof is as follows: Inductively, I can find a good minor of the transformed matrix that is not in \mathfrak{q}_{s-m} . To avoid a trivial case, I assume $[n+1, \dots, 2n] \in \mathfrak{q}_{s-m}$. Now I define

$$L_1 := \max\{c : \exists s_2, \dots, s_n \text{ with } c < s_2 < s_3 < \dots < s_n \\ \text{and } [c, s_2, \dots, s_n] \notin \mathfrak{q}_{s-m} \text{ and } [c, s_2, \dots, s_n] \text{ is good}\}$$

(then $1 \leq l_1 < n+1$) and inductively,

$$L_i := \max\{c : \exists s'_{i+1}, \dots, s'_n \text{ with } L_1 < \dots < L_{i-1} < c < s'_{i+1} < \dots < s'_n \\ \text{and } [L_1, \dots, L_{i-1}, c, s'_{i+1}, \dots, s'_n] \text{ is good} \\ \text{and } [L_1, \dots, L_{i-1}, c, s'_{i+1}, \dots, s'_n] \notin \mathfrak{q}_{s-m}\}.$$

Then $[L_1, \dots, L_n] \notin \mathfrak{q}_{s-m}$ and is good (furthermore $L_n > n$ since $[1, \dots, n] \in \mathfrak{q}_{s-m}$). I can write $[L_1, \dots, L_n] = [L_1^\alpha, \dots, L_h^\alpha; L_1^\beta, \dots, L_{n-h}^\beta]$ with

$$\{L_1^\alpha, \dots, L_h^\alpha\} \cap \{L_1^\beta, \dots, L_{n-h}^\beta\} = \emptyset. \text{ Choose } b \in (\bigcap_{i=s-m+1}^s \mathfrak{q}_i) \setminus \mathfrak{q}_{s-m} \text{ and}$$

perform a base change in A^{2n} (preserving the symmetry) by adding b times the L_i^α column of α to the L_i^α column of β , for $i = 1, \dots, h$. Then

$$[n+1, \dots, 2n]_{\text{new}} = [n+1, \dots, 2n]_{\text{old}} \pm b^h [L_1, \dots, L_n]_{\text{old}} + b\mu,$$

where $\mu \in \mathfrak{q}_{s-m}$ by the defining maximality property of the L 's. Thus $[n+1, \dots, 2n]_{\text{new}} \notin \mathfrak{q}_i$ for $i = s-m, \dots, s$, which is the inductive step. Therefore after a sequence of base changes that preserve the symmetry $\alpha\beta^t = \beta\alpha^t$ (and leave $\det(\alpha)$ unaltered) $\det(\beta)$ can be made an $A/(\det(\alpha))$ -regular element, i.e. $\det(\alpha), \det(\beta)$ is an A -regular sequence, which proves the lemma. \blacksquare

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Author's address:

MATHEMATISCHES INSTITUT DER UNIVERSITÄT BAYREUTH
UNIVERSITÄTSSTRASSE, D-95440 BAYREUTH, GERMANY

E-mail: boehning@btm8x5.mat.uni-bayreuth.de